The Spectral Flow of the Odd Signature Operator and Higher Massey Products

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0. Introduction

Given an analytic path A_t of flat connections on a principal $U(N)$-bundle P over a closed odd-dimensional manifold $M^{2\ell-1}$, how can we calculate the spectral flow of the corresponding path D_t of signature operators on an associated vector bundle $E \to M$? In this paper we give an algebraic-topological answer to this question in terms of cohomology, cup products and higher Massey products. We summarize here our technique.

Let $E_{\mathbb{C}}$ denote the complexified adjoint Lie-algebra bundle associated to P. For the purposes of this summary, we will assume that $E = E_{\mathbb{C}}$; the more general case will be dealt with in section 7. Let $d_t : \Omega^p(M; E_{\mathbb{C}}) \to \Omega^{p+1}(M; E_{\mathbb{C}})$ denote the exterior derivative corresponding to A_t for each t. At $t = 0$, we wish to calculate the dimension of $\ker(D_0)$, which gives the number of eigenvalues $\lambda_\alpha(t)$ of D_t passing through 0 at $t = 0$. Then, for each of these $\lambda_\alpha(t)$ which vanish at $t = 0$, we need to calculate the first non-vanishing derivative of $\lambda_\alpha(t)$ at $t = 0$. Because the analyticity of A_t implies that each $\lambda_\alpha(t)$ is analytic, this information will give a complete description of the spectral flow of D_t near $t = 0$.

To obtain this information, we set up a sequence of cochain complexes $\{\mathcal{G}_n^*, \delta_n\}$, for $n = 0, 1, 2, \ldots$. Note that $\mathcal{G}_n^i$ denotes the i^{th} cochain group of the n^{th} cochain complex, and the coboundary operator of this complex is δ_n. $\mathcal{G}_0^*$ is just the deRham complex of M with respect to d_0, i.e., $\mathcal{G}_0^i = \Omega^i(M; E_{\mathbb{C}})$ and $\delta_0 = d_0$. For each n, the chain groups of $\mathcal{G}_{n+1}^*$ are the cohomology groups of $\mathcal{G}_n^*$, i.e., $\mathcal{G}_{n+1}^i = H^i(\mathcal{G}_n^*, \delta_n)$. For example, $\mathcal{G}_1^i = H^i(M; E_{\mathbb{C}})_{A(0)}$.

The coboundary operator δ_n is defined by $\delta_n[u] = (-1)^{n+1}[Q_{\mathcal{M}}(u, \underbrace{a_1, \ldots, a_1}_{n})]$.

In this formula, $Q_{\mathcal{M}}$ denotes an $(n+1)$-fold higher Massey product of the cocycle u with n copies of the 1-cocycle a_1, where a_1 is the derivative $\frac{d}{dt}\big|_{t=0}A(t)$ of the path of flat connections. The Massey system $\mathcal{M}$ which we use to compute this product is required to be $A(t)$-compatible (see section 6 for the definition).

As a special case, δ_1 is just the map $H^i(M;E_{\mathbb{C}})_{A(0)} \to H^{i+1}(M;E_{\mathbb{C}})_{A(0)}$ given by the "cup product" $u \mapsto [u, a_1]$. Thus at this first level, no Massey system is needed to compute the boundary operator.

The significance of these cochain complexes $(\mathcal{G}_n^*, \delta_n)$ is as follows. The number of eigenvalues $\lambda_\alpha(t)$ of D_t which vanish to order $n-1$ at $t=0$ is given by $\sum\limits_{k \text{ even}} \dim(\mathcal{G}_n^k)$. Of those eigenvalues which vanish to order $n-1$, the difference between the number whose n^{th} derivative is positive and the number whose n^{th} derivative is negative is the signature of the Hermitian form

$$Q_n : \mathcal{G}_n^{\ell-1} \times \mathcal{G}_n^{\ell-1} \to H^{2\ell-1}(M;\mathbb{C}) = \mathbb{C}$$

given by

$$Q_n(v,w) = \begin{cases} i(-1)^{\frac{\ell+1}{2}}\delta_n v \cdot w & \text{if } \ell \text{ is odd} \\ (-1)^{\frac{\ell}{2}}\delta_n v \cdot w & \text{if } \ell \text{ is even} \end{cases}$$

The "$\cdot$" in this expression denotes the cup product constructed using the natural Hermitian structure on $u(N) \otimes \mathbb{C}$.

In essence, what this says is that if you understand the cup product and higher Massey product structures on $H^*(M;E_{\mathbb{C}})_{A(0)}$, then you understand the local behavior of those eigenvalues of the signature operator D_t which pass through 0 at $t=0$.

As one might expect, if you know the path $A(t)$ to n^{th} order, then you know the behavior of the eigenvalues to n^{th} order. For example, to understand the eigenvalues' behavior to first order, you need only know the derivative a_1 of the path $A(t)$. To construct the Massey system needed to compute δ_n, you need to know $A(t)$ to order n (again, see section 6 for details).

The first order form Q_1 was first defined in [KK1] for 3-manifolds and in [KK2] for odd-dimensional manifolds. In [FL], Farber and Levine gave a definition for the higher-order

forms $\{Q_n\}$ involving a linking form on the cohomology of M with coefficients in a power series ring. Our present definition of the Q_n has the advantage that it describes these forms in terms of cup products and higher Massey products without referring directly to power series (though power series are used in the proofs).

Our sequence $\{\mathcal{G}_n^*\}$ of chain complexes corresponds to the spectral sequence which Farber obtains in [Fa] using an analytic deformation of an elliptic complex. In our exposition, instead of using the technology of spectral sequences, we define this sequence of chain complexes "by hand" in order to show clearly that the differentials δ_n can be described as higher Massey products.

The paper is organized as follows:

In section 1, we outline our strategy for calculating derivatives of eigenvalues of an analytic path D_t of self-adjoint operators on a Hilbert space V, which involves defining a nested sequence of subspaces $V = V_0 \supseteq V_1 \supseteq V_2 \supseteq \ldots$ and Hermitian forms B_n on V_n.

In section 2, given an analytic path of operators $d_t : V \to V$ such that $d_t \circ d_t = 0$ for all t, we construct a sequence of subquotients $\mathcal{G}_n$ of V and maps $\delta_n : \mathcal{G}_n \to \mathcal{G}_n$ such that $\delta_n \circ \delta_n = 0$, $\mathcal{G}_{n+1} = \frac{\ker \delta_n}{\operatorname{im} \delta_n}$, and δ_n depends on the n^{th} order expansion of d_t.

In section 3, using an inner product on V in addition to the path $d_t : V \to V$ with $d_t \circ d_t = 0$, we construct a nested sequence of subspaces $\mathcal{H}_n$ of V such that $\mathcal{H}_n \cong \mathcal{G}_n$, and maps $\widetilde{\delta}_n : \mathcal{H}_n \to \mathcal{H}_n$ corresponding to the maps $\delta_n : \mathcal{G}_n \to \mathcal{G}_n$, such that $\mathcal{H}_n = \ker \widetilde{\delta}_n \cap \ker \widetilde{\delta}_n^*$. This gives "harmonic representatives" of the elements of $\mathcal{G}_n$.

In section 4, we show that for the operator $T_t = d_t + d_t^*$, the spaces V_n constructed in section 1 are exactly the same as the spaces $\mathcal{H}_n$ constructed in section 3.

In section 5, for a principal $U(N)$-bundle $P \to M^{2\ell-1}$, a connection A on P, and the adjoint $u(N) \otimes \mathbb{C}$-bundle $E_{\mathbb{C}} \to M$, we define the signature operator $D : \Omega^{\text{even}}(M; E_{\mathbb{C}}) \to \Omega^{\text{even}}(M; E_{\mathbb{C}})$. Under the assumption that A_t is a path of flat connections and D_t is the corresponding path of operators, we show that $V_n(D) = \bigoplus_{k \text{ even}} \mathcal{H}_n^k$, where $V_n(D)$ are the subspaces corresponding to D_t as defined in section 1 and $\mathcal{H}_n = \oplus \mathcal{H}_n^i$ are the "harmonic" subspaces defined in section 3. We also write down a formula for the

forms $B_n(D)$ on $V_n(D)$ (as defined in section 1) in terms of the operator $\widetilde{\delta}_n$ and $*$. (The dimension of $V_n(D)$ is the number of eigenvalues of D_t which vanish to order $n-1$; the n^{th} derivatives of these eigenvalues are given by the eigenvalues of $B_n(D)$.) Finally we define a "reduced version" Q_n of $B_n(D)$. Q_n has domain $\mathcal{H}_n^{\ell-1} = \mathcal{G}_n^{\ell-1}$, is defined only in terms of δ_n and cup products (hence is independent of the metric on M), and has the same signature as $B_n(D)$.

In section 6, we show that the operators δ_n are actually just higher Massey products (in the sense of Retakh) for the differential graded Lie algebra $\Omega^*(M; E_{\mathbb{C}})$.

Finally, in section 7, we indicate how to handle the case of the odd signature operator on a general bundle $E \to M$. This involves "hybrid" Massey products taking into account the action of the Lie algebra bundle $E_{\mathbb{C}}$ on E.

1. Derivatives of vanishing eigenvalues of a path of self-adjoint operators

Let $D_t : V \to V$ be an analytic path of closed, self-adjoint operators on the complex Hilbert space V. We will assume that each D_t has compact resolvent, so that the spectrum of D_t is a discrete subset of $\mathbb{R}$ and each eigenvalue has finite multiplicity.

We recall from [Ka] that the path D_t is called *analytic* if there exists a Hilbert space W and a path Q_t in $Bd(W, V)$ so that Q_t is analytic (with respect to the operator norm topology), the domain of D_t is the range of Q_t, and the composite $D_t \circ Q_t$ is an analytic path in $Bd(W, V)$.

Note. The way this set-up will occur later in the paper is that V will be a space of sections of a Hermitian vector bundle over a closed Riemannian manifold and D_t will be an analytic path of first order elliptic operators which are self-adjoint with respect to the L^2 inner product. For details, see section 5. Of course, for the simplest example one may think of V as finite dimensional and D_t as an analytic path of Hermitian matrices.

By [Ka], there is an analytic family $\{\varphi_\alpha(t)\}_{\alpha\in A}$ of orthonormal Hilbert bases for V and a collection of analytic real valued functions $\{\lambda_\alpha(t)\}_{\alpha\in A}$ such that for all $t\in(-\varepsilon,\varepsilon)$,

$$D_t\,\varphi_\alpha(t)=\lambda_\alpha(t)\,\varphi_\alpha(t).$$

We wish to calculate the first non-vanishing derivative of $\lambda_\alpha(t)$ at $t=0$ for those λ_α's satisfying $\lambda_\alpha(0)=0$. We accomplish this by defining and diagonalizing a sequence of bilinear forms on a nested sequence of subspaces of V.

First, define the subsets $A_0, A_1, A_2, \ldots$ of A by

$$A_0 = A$$

$$A_n=\left\{\alpha\in A : \left.\frac{d^i}{dt^i}\right|_{t=0}\lambda_\alpha(t)=0 \text{ for all } i<n\right\}$$

Define subspaces $V_n\subseteq V$ by

$$V_n=\operatorname*{span}_{\alpha\in A_n}\{\varphi_\alpha(0)\}$$

For each n, define a Hermitian bilinear form $B_n : V_n\times V_n\to\mathbb{C}$ as follows.

Given $v,w\in V_n$, write $v=\sum_{\alpha\in A_n} c_\alpha\,\varphi_\alpha(0)$. Define

$$B_n(v,w)=\left\langle \left.\frac{d^n}{dt^n}\right|_{t=0} D_t\sum_{\alpha\in A_n} c_\alpha\,\varphi_\alpha(t), w\right\rangle .$$

From this definition it follows immediately that

$$B_n(\varphi_\alpha(0),\varphi_\beta(0))=\lambda_\alpha^{(n)}(0)\delta_{\alpha\beta}.$$

Hence B_n is Hermitian and its eigenvalues are the n^{th} derivatives of those eigenvalues of D_t which vanish to order $n-1$. It also follows that V_{n+1} is the subspace on which B_n is degenerate.

2. A sequence of chain complexes

Let V be a vector space and let $d_t : V \to V$ be an analytic path of linear transformations such that $d_t \circ d_t = 0$ for all t near 0. To avoid any technicalities, we will assume that $d_t = d_0 + \sum_{i=1}^{\infty} A_i t^i$ where $\sum_{i=1}^{\infty} A_i t^i$ is a convergent power series in $Bd(V, V)$. and d_0 is some closed linear operator. The domain of d_t is taken to be the same for all t. (In the main example to keep in mind, V will be the L^2 closure of $\oplus_n V_n$ where (V_n, d_t) is an elliptic complex over a closed smooth manifold. In particular, $\ker d_0 / \operatorname{im} d_0$ will be finite dimensional.)

We will define inductively a sequence $\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \ldots$ of vector spaces and $\delta_n : \mathcal{G}_n \to \mathcal{G}_n$ of linear transformations as follows.

First, let $\mathcal{G}_0 = V$ and $\delta_0 = d_0$.

Inductively, let $\mathcal{G}_n = \ker(\delta_{n-1}) / \operatorname{im}(\delta_{n-1})$.

To define δ_n, note that $\mathcal{G}_n$ is a subquotient of V. Hence an element of $\mathcal{G}_n$ may be written as $[u]$ where $u \in V$. Define

$$\delta_n([u]) = \left[\left. \frac{d^n}{dt^n} \right|_{t=0} d_t(u + u_1 t + \ldots + u_{n-1} t^{n-1}) \right]$$

where $u_1, \ldots, u_{n-1} \in V$ are chosen so that for all $i < n$,

$$\left. \frac{d^i}{dt^i} \right|_{t=0} d_t(u + u_1 t + \ldots + u_{n-1} t^{n-1}) = 0.$$

Theorem 2.1. *The spaces $\mathcal{G}_n$ and operators $\delta_n : \mathcal{G}_n \to \mathcal{G}_n$ are well-defined and $\delta_n \circ \delta_n = 0$ for all n.*

The rest of the section is devoted to proving this theorem.

Proof. Since d_t is analytic we may write $d_t = d + A_1 t + A_2 t^2 + \ldots$ where each A_i is linear and $d = d_0$. In this notation, if we apply d_t to a formal power series in t with coefficients in V, we obtain another such formal power series. With this in mind, we define a sequence

$\{T_n\}_{n=0}^{\infty}$ of linear relations on V as follows: T_0 is the function d_0. Define uT_nv if and only if there exist $u_1, \ldots, u_{n-1} \in V$ such that $d_t(u + u_1t + \ldots + u_{n-1}t^{n-1}) = vt^n + w(t)t^{n+1}$ for some $w(t) \in V[[t]]$. As usual, we define

$$\text{Domain}(T_n) = \{u \in V : uT_nv \text{ for some } v \in V\}$$

$$\text{Range}(T_n) = \{v \in V : uT_nv \text{ for some } u \in V\}$$

We now state four lemmas concerning these relations.

Lemma 2.2. *Suppose $n \geq 1$, $k \geq 0$ and $u, v \in V$. Then $uT_nv \Rightarrow vT_k0$.*

Lemma 2.3. *Suppose $n \geq 1$. Then $v \in$ Range$(T_n) \Leftrightarrow 0T_{n+1}v$.*

Lemma 2.4. *Suppose $v = du$. Then for all $n \geq 0$, there exists $w_n \in V$ such that $vT_n\, dw_n$.*

Lemma 2.5.

(a) $u \in$ Domain$(T_1) \Leftrightarrow du = 0$.

(b) If $n \geq 2$, then $u \in$ Domain$(T_n) \Leftrightarrow uT_{n-1}\, dw$ for some $w \in V$.

Proofs.

(2.2) uT_nv implies we may choose $u_1, \ldots, u_{n-1} \in V$ such that $d_t(u + u_1t + \ldots + u_{n-1}t^{n-1}) = vt^n + w(t)t^{n+1} = (v + w(t)t)t^n$. Since $d_t^2 = 0$, it follows that $d_t(v + w(t)t) = 0$ which implies that vT_k0 for all $k \geq 0$.

(2.3)($\Rightarrow$) Since uT_nv, there exist $u_1, \ldots, u_{n-1}$ such that $d_t(u + u_1t + \ldots + u_{n-1}t^{n-1}) = vt^n + w(t)t^{n+1}$. Multiplying both sides by t immediately implies $0T_{n+1}v$.

($\Leftarrow$) Just reverse the above argument.

(2.4) Define $v_i = A_iu$ for $i \geq 1$, so $d_tu = (v + v_1t + v_2t^2 + \ldots)$. Since $d_t^2 = 0$, $d_t(v + v_1t + v_2t^2 + \ldots) = 0$ which implies $d_t(v + v_1t + \ldots + v_{n-1}t^{n-1}) = -d_t(v_nt^n + \ldots)$ which we

can write

$$d_t(v + v_1 t + \ldots + v_{n-1} t^{n-1}) = d(-v_n) t^n + w(t) t^{n+1}.$$

Hence we may let $w_n = -v_n$ and the lemma follows.

(2.5) (a) There exists $v \in V$ such that $uT_1 v$.

$\Leftrightarrow$ There exists $v \in V$ such that $d_t u = vt + w(t)t^2$.

$\Leftrightarrow$ $du = 0$.

(b)($\Rightarrow$) $u \in$ Domain(T_n) implies there exist $u_1, \ldots, u_{n-1}$ such that

$$d_t(u + u_1 t + \ldots + u_{n-1} t^{n-1}) = w(t) t^n$$

for some $w(t) \in V[[t]]$. It follows that

$$\begin{aligned} d_t(u + u_1 t + \ldots + u_{n-2} t^{n-2}) &= -d_t u_{n-1} t^{n-1} + y(t) t^n \\ &= d(-u_{n-1}) t^{n-1} + \widetilde{y}(t) t^n \end{aligned}$$

where $\widetilde{y}(t) \in V[[t]]$. So $uT_{n-1}\, d(-u_{n-1})$.

($\Leftarrow$) Suppose $uT_{n-1}\, dw$. Then

$$d_t(u + u_1 t + \ldots + u_{n-2} t^{n-2}) = dw t^{n-1} + y(t) t^n.$$

Let $u_{n-1} = -w$. Then

$$\begin{aligned} d_t(u + u_1 t + \ldots + u_{n-1} t^{n-1}) &= -d_t w t^{n-1} + dw t^{n-1} + y(t) t^n \\ &= \widetilde{y}(t) t^n. \end{aligned}$$

So $u \in$ Domain(T_n).

Having proven these lemmas, we make the following definitions:

Definition. Let $\mathcal{Z}_n$ and $\mathcal{B}_n$ denote the following subspaces of V for $n = 1, 2, 3, \ldots$.

$$\mathcal{Z}_1 = \ker(d)$$

$$\mathcal{B}_1 = \mathrm{im}(d).$$

For $n \geq 2$,

$$\mathcal{Z}_n = \{u \in V : uT_{n-1}\, dw \text{ for some } w\}$$

$$\mathcal{B}_n = \text{ Range}(T_{n-1}) + dV.$$

Proposition 2.6. *For all* $n \geq 1$,

(a) $\mathcal{B}_n \subseteq \mathcal{Z}_n$.

(b) $\mathcal{Z}_{n+1} \subseteq \mathcal{Z}_n$.

(c) $\mathcal{B}_n \subseteq \mathcal{B}_{n+1}$.

This proposition follows easily from the above lemmas and we omit the proof.

Define $G_n = \mathcal{Z}_n \,/\, \mathcal{B}_n$.

Proposition 2.7. T_n *induces a well-defined linear operator* $\tau_n : G_n \to G_n$ *and* $\tau_n \circ \tau_n = 0$.

Proof. $(n = 1)$. $u \in \mathcal{Z}_1 \Rightarrow du = 0 \Rightarrow u \in$ Domain(T_1) by Lemma 2.5. If $u \in \mathcal{Z}_1$ and uT_1v then, by Lemma 2.2, $v \in \mathcal{Z}_1$.

To see that T_1 induces a well-defined $\tau_1 : G_1 \to G_1$, we note first that there are no choices involved in defining T_1, and then note that by Lemma 2.4, $T_1(\mathcal{B}_1) \subseteq \mathcal{B}_1$.

$(n \geq 2)$. If $u \in \mathcal{Z}_n$ then, by Lemma 2.5, $u \in$ Domain(T_n). By Lemma 2.2, if uT_nv then $v \in \mathcal{Z}_n$.

If $u \in \mathcal{B}_n$, then we can write $u = u_1 + u_2$ where there exists an x such that $xT_{n-1}u_1$ and there exists a y such that $dy = u_2$. It follows by Lemmas 2.2 and 2.4 that uT_nv where $v \in \mathcal{B}_n$. If it is also true that uT_nv', then $0T_n(v - v')$. Hence $v - v' \in$ Range(T_{n-1}) (by Lemma 2.3) so v' is also in $\mathcal{B}_n$. Hence $\tau_n : G_n \to G_n$ is well-defined.

Lemma 2.2 implies that $\tau_n \circ \tau_n = 0$ for all n.

Proposition 2.8. $G_{n+1} = \ker \tau_n / \operatorname{im} \tau_n$.

Proof. $\tau_n : G_n \to G_n$ where $G_n = \mathcal{Z}_n / \mathcal{B}_n$. It is true that $\ker \tau_n = \mathcal{Z}_{n+1} / \mathcal{B}_n$ and $\operatorname{im}(\tau_n) = \mathcal{B}_{n+1} / \mathcal{B}_n$. We will show that: $\ker \tau_n \subseteq \mathcal{Z}_{n+1} / \mathcal{B}_n$, which is the only non-trivial part.

Suppose $u \in \mathcal{Z}_n$ and $\tau_n[u] = 0$. Then uT_nv where $v \in \mathcal{B}_n$. Write $v = v_1 + v_2$ where $v_1 \in$ Range T_{n-1} and $v_2 \in dV$. Since $v_1 \in$ Range T_{n-1}, Lemma 2.3 implies that $0T_nv_1$, so uT_nv_2. Hence $u \in \mathcal{Z}_{n+1}$, and $\ker \tau_n \subseteq \mathcal{Z}_{n+1} / \mathcal{B}_n$. ∎

Letting $\mathcal{G}_n = G_n$ and $\delta_n = n!\tau_n$, we have now proven theorem 2.1, since δ_n and $\mathcal{G}_n$ satisfy the formula and inductive definition given at the beginning of this section.

Some of the results of this section can also be described in terms of a spectral sequence; this is essentially the content of Theorem 6.1 of [Fa]. We give a brief alternative description here for completeness. (Note that for much of what follows in our paper, it will still be important to have the explicit description of τ_n which we have given above.)

Let $M = V[[t]]$ denote the formal power series with coefficients in V, and filter M by $F^p = t^pM$. The differential $d_t : M \to M$ preserves the filtration and thus one obtains a spectral sequence. In terms of an exact couple one takes $D_1 = \oplus D_1^p$ where $D_1^p = H^*(F^p; d_t)$ and $E_1 = \oplus E_1^p$ where $E_1^p = H^*(F^{p-1}/F^p; d_t)$. Then

$$\begin{array}{ccc} D_1 & \rightarrow & D_1 \\ & \nwarrow \quad \swarrow & \\ & E_1 & \end{array}$$

is an exact couple. One identifies the higher E_r by taking $E_r^p = Z_r^p / B_r^p$ where

$$Z_r^p = k^{-1}(\mathrm{Im} i^{r-1} : D_1^{p+r} \to D_1^{p+1})$$

and

$$B_r^p = j(\ker i^{r-1} : D_1^p \to D_1^{p-r+1}).$$

(See [Mc].) It is then a routine exercise to show inductively that $G_n \cong E_n^p$ for any $p \geq n-1$, and that $\tau_n : G_n \to G_n$ coincides with the induced differential $d_n : E_n^p \to E_n^{p-n}$ for $p \geq 2n-1$.

3. Hodge Theory

Assume we are in the situation of the last chapter; i.e., $d_t : V \to V$ is an analytic path of linear transformation such that $d_t^2 = 0$. Assume in addition that V has a Hermitian inner product $\langle \quad , \quad \rangle$. For each t, let $d_t^* : V \to V$ denote the adjoint of d_t. Clearly $d_t^* \circ d_t^* = 0$.

We assume furthermore that d_t^* is a closed operator with the same domain as d_t. Recall that $d_t = d_0 + A(t)$ where $A(t)$ is an analytic path in $Bd(V,V)$. Thus $d_t^* = d_0^* + A(t)^*$ is an analytic path.

Recall from section 2 that $\mathcal{Z}_1 = \ker d$ and $\mathcal{Z}_n = \{u \in V : uT_{n-1}\, dw \text{ for some } w \in V\}$ for $n \geq 2$. By Lemma 2.5 in section 2, $\mathcal{Z}_n =$ Domain T_n. Hence an alternative description of $\mathcal{Z}_n$ (for $n \geq 1$) is

$$\{u \in V : \exists\, u_1, \ldots, u_{n-1} \in V \text{ such that}$$

$$\left.\frac{d^k}{dt^k}\right|_{t=0} d_t(u + tu_1 + \ldots + t^{n-1}u_{n-1}) = 0 \text{ for all } k < n\}$$

$$= \{u \in V : \exists\, u_1, \ldots, u_{n-1} \in V \text{ such that}$$

$$d_t(u + tu_1 + \ldots + t^{n-1}u_{n-1}) = v(t)t^n \text{ for some } v(t) \in V[[t]]\}.$$

Define $\mathcal{Z}_n^* \subseteq V$ analogously using d_t^*:

$$\mathcal{Z}_1^* = \ker d^*$$

$$\mathcal{Z}_n^* = \{u \in V : \exists\, u_1, \ldots, u_{n-1} \in V \text{ such that}$$

$$d_t^*(u + tu_1 + \ldots + t^{n-1}u_{n-1}) = v(t)t^n \text{ for some } v(t) \in V[[t]]\}.$$

Define $\mathcal{H}_n = \mathcal{Z}_n \cap \mathcal{Z}_n^*$. Note that $\mathcal{Z}_1 \supseteq \mathcal{Z}_2 \supseteq \mathcal{Z}_3 \supseteq \ldots$ and $\mathcal{Z}_1^* \supseteq \mathcal{Z}_2^* \supseteq \mathcal{Z}_3^* \supseteq \ldots$ are nested sequences, hence so is $\mathcal{H}_1 \supseteq \mathcal{H}_2 \supseteq \mathcal{H}_3 \supseteq \ldots$.

Lemma 3.1. $\mathcal{Z}_n^* \subseteq \mathcal{B}_n^\perp$.

Proof. Let $u \in \mathcal{Z}_n^*$; i.e., assume that there exist $u_1, \ldots, u_{n-1} \in V$ such that $d_t^*(u + tu_1 + \ldots + t^{n-1}u_{n-1}) = q(t)t^n$ for some $q(t) \in V[[t]]$. We need to show $u \perp dV$ and $u \perp$ Range(T_{n-1}). Since $u \in \mathcal{Z}_n^* \subseteq \mathcal{Z}_1^* =$ ker d^* it follows that $u \perp dV$. Now let $v \in$ Range(T_{n-1}); i.e., assume there exist $w, w_1, \ldots, w_{n-2}$ such that

$$d_t(w + tw_1 + \ldots + t^{n-2}w_{n-2}) = vt^{n-1} + z(t)t^n.$$

Clearly

$$\begin{aligned}
\langle u, v\rangle t^{n-1} + (\quad)t^n &= \langle u + tu_1 + \ldots + t^{n-1}u_{n-1}, vt^{n-1} + z(t)t^n\rangle \\
&= \langle u + tu_1 + \ldots + t^{n-1}u_{n-1}, d_t(w + tw_1 + \ldots + t^{n-2}w_{n-2})\rangle \\
&= \langle q(t)t^n, w + tw_1 + \ldots + t^{n-2}w_{n-2}\rangle \\
&= \langle q(t), w + tw_1 + \ldots, t^{n-2}w_{n-2}\rangle t^n.
\end{aligned}$$

∎

So $\langle u, v\rangle = 0$. Since $\mathcal{Z}_n^* \subseteq \mathcal{B}_n^\perp$, it follows that the map

$$\Phi_n : \mathcal{Z}_n \cap \mathcal{Z}_n^* \to \frac{\mathcal{Z}_n}{\mathcal{B}_n}$$

is injective. Recall from section 2 that $\mathcal{Z}_n =$ Domain(T_n) and Range$(T_n) \subseteq \mathcal{Z}_n$. Also the indeterminancy of T_n is contained in $\mathcal{B}_n$. It follows that we have a well-defined homomorphism $\widetilde{\delta}_n : \mathcal{Z}_n \cap \mathcal{Z}_n^* \to \mathcal{Z}_n \cap \mathcal{Z}_n^*$ defined by $\widetilde{\delta}_n = pr_{\mathcal{H}_n} \circ T_n$. Explicitly,

$$\widetilde{\delta}_n(u) = pr_{\mathcal{H}_n}\left(\left.\frac{d^n}{dt^n}\right|_{t=0} d_t(u + tu_1 + \ldots, t^{n-1}u_{n-1})\right)$$

where $u_1, \ldots, u_{n-1}$ are chosen so that

$$\left.\frac{d^k}{dt^k}\right|_{t=0} d_t(u + tu_1 + \ldots, t^{n-1}u_{n-1}) = 0$$

for all $k < n$. By symmetry, the homomorphism $\widetilde{\delta}_n^* : \mathcal{Z}_n \cap \mathcal{Z}_n^* \to \mathcal{Z}_n \cap \mathcal{Z}_n^*$ defined by

$$\widetilde{\delta}_n^* u = pr_{\mathcal{H}_n} \frac{d^n}{dt^n}\bigg|_{t=0} d_t^*(u + tu_1 + \ldots + t^{n-1}u_{n-1}) = 0,$$

where $u_1, \ldots, u_{n-1}$ are chosen so that

$$\frac{d^k}{dt^k}\bigg|_{t=0} d_t^*(u + tu_1 + \ldots + t^{n-1}u_{n-1}) = 0$$

for all $k < n$, is also well-defined.

We now justify the notation "$\widetilde{\delta}_n^*$."

Proposition 3.2. *$\widetilde{\delta}_n$ and $\widetilde{\delta}_n^*$ are adjoint on $\mathcal{H}_n$.*

Proof. Let $u, v \in \mathcal{H}_n$. Choose $u_1, \ldots, u_{n-1}$ so that

$$\frac{d^k}{dt^k}\bigg|_{t=0} d_t(u + tu_1 + \ldots + t^{n-1}u_{n-1}) = 0$$

for all $k < n$ and choose $v_1, \ldots, v_{n-1}$ so that

$$\frac{d^k}{dt^k}\bigg|_{t=0} d_t^*(v + tu_1 + \ldots + t^{n-1}v_{n-1}) = 0$$

for all $k < n$. Note that

$$\langle d_t(u + tu_1 + \ldots + t^{n-1}u_{n-1}), (v + tu_1 + \ldots + t^{n-1}v_{n-1})\rangle$$

$$-\langle (u + tu_1 + \ldots + t^{n-1}u_{n-1}), d_t^*(v + tu_1 + \ldots + t^{n-1}v_{n-1})\rangle = 0.$$

Differentiating this expression n times at $t = 0$ yields

$$\left\langle \frac{d^n}{dt^n}\bigg|_{t=0} d_t(u + tu_1 + \ldots + t^{n-1}u_{n-1}), v \right\rangle = \left\langle u, \frac{d^n}{dt^n}\bigg|_{t=0} d_t^*(v + tv_1 + \ldots + t^{n-1}v_{n-1}) \right\rangle.$$

Since u, $v \in \mathcal{H}_n$, inserting $pr_{\mathcal{H}_n}$ in the appropriate places doesn't alter the values of these inner products. Hence

$$\langle \widetilde{\delta}_n u, v\rangle = \langle u, \widetilde{\delta}_n^* v\rangle.$$

$\blacksquare$

Theorem 3.3. *For all $n \geq 1$,*

$$\Phi_n : \mathcal{Z}_n \cap \mathcal{Z}_n^* \to \frac{\mathcal{Z}_n}{\mathcal{B}_n} = \mathcal{G}_n$$

is an isomorphism and $\Phi_n \delta_n \Phi_n^{-1} = \widetilde{\delta}_n$.

Proof. We prove the theorem by induction on n.

First consider the case $n = 1$, In this case $\mathcal{Z}_1^* = \ker d^* = \operatorname{im}(d)^\perp = \mathcal{B}_1^\perp$, so

$$\Phi_1 : \mathcal{Z}_1 \cap \mathcal{Z}_1^* \to \frac{\mathcal{Z}_1}{\mathcal{B}_1}$$

is an isomorphism. The equation $\Phi_1 \delta_1 \Phi_1^{-1} = \widetilde{\delta}_1$ follows immediately from the definitions.

For the inductive step, assume we've proven that $\Phi_n : \mathcal{Z}_n \cap \mathcal{Z}_n^* \to \mathcal{Z}_n \,/\, \mathcal{B}_n$ is an isomorphism and $\Phi_n \delta_n \Phi_n^{-1} = \widetilde{\delta}_n$. Note that (by section 2),

$$\frac{\mathcal{Z}_{n+1}}{\mathcal{B}_{n+1}} = \frac{\ker \delta_n}{\operatorname{im} \delta_n} \cong \frac{\ker \widetilde{\delta}_n}{\operatorname{im} \widetilde{\delta}_n}$$

$$\cong \ker \widetilde{\delta}_n \cap \ker \widetilde{\delta}_n^* .$$

(The last isomorphism follows because $\widetilde{\delta}_n \circ \widetilde{\delta}_n = 0$ since $\widetilde{\delta}_n = \Phi_n \circ \delta_n \circ \Phi_n^{-1}$.)

All of these isomorphisms are induced by inclusion.

Lemma. $\ker(\widetilde{\delta}_n) \cap \ker(\widetilde{\delta}_n^*) = \mathcal{Z}_{n+1} \cap \mathcal{Z}_{n+1}^*$.

Proof. We begin by showing that $\mathcal{Z}_{n+1} \cap \mathcal{H}_n \subseteq \ker(\widetilde{\delta}_n)$. If $u \in \mathcal{Z}_{n+1} \cap \mathcal{H}_n$, we can choose $u_1, \ldots, u_n$ so that $d_t(u + tu_1 + \ldots + t^n u_n) = z(t)t^{n+1}$. Hence

$$d_t(u + tu_1 + \ldots + t^{n-1}u_{n-1}) = -d_t u_n t^n + z(t)t^{n+1}$$

$$= -du_n t^n + \widetilde{z}(t)t^{n+1}.$$

So $\widetilde{\delta}_n(u) = pr_{\mathcal{H}_n}(-n!\,du_n) = 0$ since $du_n \in \mathcal{B}_n \subseteq (\mathcal{Z}_n^*)^{\perp}$.

By symmetry we also conclude that $\mathcal{Z}_{n+1}^* \cap \mathcal{H}_n \subseteq \ker(\widetilde{\delta}_n^*)$; hence

$$\mathcal{Z}_{n+1} \cap \mathcal{Z}_{n+1}^* = (\mathcal{Z}_{n+1} \cap \mathcal{H}_n) \cap (\mathcal{Z}_{n+1}^* \cap \mathcal{H}_n) \subseteq \ker(\widetilde{\delta}_n) \cap \ker(\widetilde{\delta}_n^*).$$

We now show that $\ker(\widetilde{\delta}_n) \subseteq \mathcal{Z}_{n+1}$.

If $u \in \ker(\widetilde{\delta}_n)$, we can choose $u_1, \ldots, u_{n-1}$ so that

$$d_t(u + tu_1 + \ldots + t^{n-1}u_{n-1}) = vt^n + (\quad)t^{n+1} \tag{$*$}$$

where $pr_{\mathcal{H}_n}(v) = 0$. Since $\mathcal{Z}_n^* \subseteq \mathcal{B}_n^{\perp}$ and by the inductive assumption that

$$\Phi_n : \mathcal{Z}_n \cap \mathcal{Z}_n^* \to \frac{\mathcal{Z}_n}{\mathcal{B}_n}$$

is an isomorphism, it follows that $v \in \mathcal{B}_n$. (The inductive assumption implies that $\mathcal{Z}_n$ is the orthogonal direct sum of $\mathcal{H}_n$ and $\mathcal{B}_n$.) Hence $v = \widetilde{v} + dw$ where $\widetilde{v} \in$ Range T_{n-1}. By Lemma 2.3 in section 2, $\widetilde{v} \in$ Range $T_{n-1} \Rightarrow 0T_n\,\widetilde{v}$. Since (*) above can be written uT_nv, it follows by the linearity of the relation that $uT_n\,dw$ which means $u \in \mathcal{Z}_{n+1}$, proving that $\ker(\widetilde{\delta}_n) \subseteq \mathcal{Z}_{n+1}$. By symmetry we also conclude that $\ker(\widetilde{\delta}_n^*) \subset \mathcal{Z}_{n+1}^*$, hence $\ker(\widetilde{\delta}_n) \cap \ker(\widetilde{\delta}_n^*) \subseteq \mathcal{Z}_{n+1} \cap \mathcal{Z}_{n+1}^*$, completing the proof that $\ker(\widetilde{\delta}_n) \cap \ker(\widetilde{\delta}_n^*) = \mathcal{Z}_{n+1} \cap \mathcal{Z}_{n+1}^*$. ∎

Since we've already shown that

$$\frac{\mathcal{Z}_{n+1}}{\mathcal{B}_{n+1}} \cong \ker\,\widetilde{\delta}_n \cap \ker\,\widetilde{\delta}_n^*,$$

the proof that

$$\mathcal{H}_{n+1} = \mathcal{Z}_{n+1} \cap \mathcal{Z}_{n+1}^* \cong \frac{\mathcal{Z}_{n+1}}{\mathcal{B}_{n+1}}$$

is now complete. It immediately follows that there is an orthogonal decomposition $\mathcal{Z}_{n+1} = \mathcal{H}_{n+1} \oplus \mathcal{B}_{n+1}$, and from the definitions of $\delta_n, \widetilde{\delta}_n$ we obtain

$$\Phi_{n+1}\delta_{n+1}\Phi_{n+1}^{-1} = \widetilde{\delta}_{n+1},$$

completing the induction step and the theorem. ∎

4. The operator $d + d^*$

We continue with the assumptions of the previous section. Thus V is a Hilbert space, $d_t : V \to V$ is an analytic path of linear operators, and $d_t \circ d_t = 0$ for all t near 0. The adjoint d_t^* has the same domain as d_t (which has the same domain as d_0).

Define $T_t = d_t + d_t^*$ with domain equal to the domain of d_t. This is clearly an analytic path of self-adjoint operators. For the remainder of this section, we also assume that for each t the operator T_t has compact resolvent. (This is true, for example, if d_t is a path of exterior derivatives on the de Rham complex of a compact manifold since in this case $d_t + d_t^*$ is an elliptic operator.) It follows that we may apply the apparatus of section 1 to the operator T_t to obtain subspaces V_n. The goal of this section is to show that the subspaces V_n (as defined in section 1) coincide with the subspaces $\mathcal{H}_n$ (as defined in section 3).

Recall from section 1 that we have an analytic path $\{\varphi_\alpha(t)\}_{\alpha \in A}$ of orthonormal bases of V and analytic paths $\{\lambda_\alpha(t)\}$ of real eigenvalues satisfying

$$T_t\, \varphi_\alpha(t) = \lambda_\alpha(t)\, \varphi_\alpha(t).$$

Recall that for $n = 0, 1, 2, \ldots$

$$A_n = \left\{ \alpha \in A : \left.\frac{d^k}{dt^k}\right|_{t=0} \lambda_\alpha(t) = 0 \text{ for all } k < n \right\}$$

and

$$V_n = \operatorname*{span}_{\alpha \in A_n} \{\varphi_\alpha(0)\}.$$

Theorem 4.1. $V_n = \mathcal{H}_n$.

Lemma 4.2. *Let $u(t)$ and $v(t)$ be two smooth paths in V such that for all t, $\langle u(t), v(t)\rangle = 0$. Let $n \geq 0$. Then*

$$\left.\frac{d^k}{dt^k}\right|_{t=0} (u(t) + v(t)) = 0$$

for all $k \leq n$ if and only if

$$\left.\frac{d^k}{dt^k}\right|_{t=0} u(t) = 0 \quad \text{and} \quad \left.\frac{d^k}{dt^k}\right|_{t=0} v(t) = 0$$

for all $k \leq n$.

Proof of Lemma. The direction ($\Leftarrow$) is obvious. We prove ($\Rightarrow$) by induction.

For $n = 0$, $u(0) + v(0) = 0$ implies $\langle u(0), u(0)\rangle = \langle u(0), u(0) + v(0)\rangle = 0$, so $u(0) = 0$ and similarly $v(0) = 0$.

Now assume the lemma is true for n, and suppose

$$\left.\frac{d^k}{dt^k}\right|_{t=0} (u(t) + v(t)) = 0$$

for all $k \leq n + 1$. By the induction assumption,

$$\left.\frac{d^k}{dt^k}\right|_{t=0} u(t) = 0 \quad \text{and} \quad \left.\frac{d^k}{dt^k}\right|_{t=0} v(t) = 0$$

for all $k \leq n$.

Differentiate the equation

$$\langle u(t), v(t)\rangle = 0$$

$2n+2$ times at $t = 0$. After cancelling all terms which vanish by the induction assumption, we are left with

$$\binom{2n+2}{n+1} \left\langle \left.\frac{d^{n+1}}{dt^{n+1}}\right|_{t=0} u(t), \left.\frac{d^{n+1}}{dt^{n+1}}\right|_{t=0} v(t) \right\rangle = 0.$$

Since

$$\left.\frac{d^{n+1}}{dt^{n+1}}\right|_{t=0} u(t) + \left.\frac{d^{n+1}}{dt^{n+1}}\right|_{t=0} v(t) = 0,$$

it follows that each of these terms are 0, finishing the proof of the lemma. ∎

Proof of Theorem 4.1. First we show $V_n \subseteq \mathcal{H}_n$. Let $v \in V_n$ and write $v = \sum_{\alpha \in A_n} c_\alpha \varphi_\alpha(0)$. Let $v(t) = \sum_{\alpha \in A_n} c_\alpha \varphi_\alpha(t)$. By definition of A_n,

$$\left.\frac{d^k}{dt^k}\right|_{t=0} T_t v(t) = 0$$

for all $k < n$. Since $T_t v(t) = d_t v(t) + d_t^* v(t)$ and $\mathrm{im}(d_t) \perp \mathrm{im}(d_t^*)$ for all t, the previous lemma implies that

$$\left.\frac{d^k}{dt^k}\right|_{t=0} d_t v(t) = 0 \quad \text{and} \quad \left.\frac{d^k}{dt^k}\right|_{t=0} d_t^* v(t) = 0$$

for all $k < n$. Hence $v \in \mathcal{Z}_n \cap \mathcal{Z}_n^* = \mathcal{H}_n$.

We will now show by induction that $\mathcal{H}_n \subseteq V_n$ for all $n \geq 1$. For $n = 1$, clearly $\mathcal{H}_1 = \ker d \cap \ker d^* = \ker(d + d^*) = V_1$. Assume that $\mathcal{H}_n = V_n$; i.e., $\mathcal{Z}_n \cap \mathcal{Z}_n^* = V_n$. Let $v \in \mathcal{H}_{n+1}$. Since $\mathcal{H}_{n+1} \subseteq \mathcal{H}_n = V_n$, we can write $v = \sum_{\alpha \in A_n} c_\alpha \varphi_\alpha(0)$. Define $v(t) = \sum_{\alpha \in A_n} c_\alpha \varphi_\alpha(t)$. As in the first part of this proof, we may conclude by Lemma 4.2 that

$$\left.\frac{d^k}{dt^k}\right|_{t=0} d_t v(t) = 0 \quad \text{and} \quad \left.\frac{d^k}{dt^k}\right|_{t=0} d_t^* v(t) = 0$$

for all $k < n$. Hence we may use (the degree $n-1$ truncation of) $v(t)$ to compute $\widetilde{\delta}_n(v)$ and $\widetilde{\delta}_n^*(v)$. Keep in mind that since $v \in \mathcal{H}_{n+1}$, $\widetilde{\delta}_n(v) = \widetilde{\delta}_n^*(v) = 0$. Let $\beta \in A_n - A_{n+1}$; i.e., $\lambda_\beta^{(k)}(0) = 0$ for $k < n$ but $\lambda_\beta^{(n)}(0) \neq 0$. Compute

$$\begin{aligned}
\langle v, \lambda_\beta^{(n)}(0)\, \varphi_\beta(0)\rangle &= \left\langle v, \left.\frac{d^n}{dt^n}\right|_{t=0} T_t\, \varphi_\beta(t) \right\rangle \\
&= \left.\frac{d^n}{dt^n}\right|_{t=0} \langle v(t), T_t\, \varphi_\beta(t)\rangle \\
&= \left.\frac{d^n}{dt^n}\right|_{t=0} \langle T_t v(t), \varphi_\beta(t)\rangle \\
&= \left.\frac{d^n}{dt^n}\right|_{t=0} \langle d_t v(t) + d_t^* v(t), \varphi_\beta(t)\rangle \\
&= \langle \widetilde{\delta}_n(v) + \widetilde{\delta}_n^*(v), \varphi_\beta(0)\rangle \\
&= 0
\end{aligned}$$

Hence for all $\beta \in A_n - A_{n+1}$, $\langle v, \varphi_\beta(0)\rangle = 0$ and we conclude that $v \in V_{n+1}$, completing the proof of the theorem. ■

Note. (1) A more direct way to prove this theorem would be to prove simply that $\mathcal{Z}_n^* = \mathcal{B}_n^\perp$. We were able to do this for $n = 1$ and 2, but we are not sure if it is true for higher n.

(2) Since for each n, $V_n = \mathcal{H}_n \cong \mathcal{G}_n$, we have shown that $\dim(V_n)$ does not depend on the choice of inner product, but only on the path d_t.

5. The odd signature operator

In this section we recall the definition of the signature operator on an odd-dimensional manifold and apply the material of the previous section to obtain information about the derivatives of its eigenvalues.

Let M be a closed oriented Riemannian manifold of dimension $2\ell - 1$, $P \to M$ a principal $U(N)$-bundle and $E \to M$ the $u(N)$-bundle associated to P by the adjoint representation $U(N) \to O(u(N))$. Denote by $E_{\mathbb{C}}$ the complexification $E \otimes \mathbb{C}$. If A is a connection on P, let

$$d_A : \Omega^p(M; E_{\mathbb{C}}) \to \Omega^{p+1}(M; E_{\mathbb{C}})$$

denote the corresponding exterior derivative. The standard inner product on $u(N)$ extends to a Hermitian inner product on $u(N) \otimes \mathbb{C}$ given by $\langle a \otimes z, b \otimes w \rangle = -tr(ab) \otimes z\,\overline{w}$. We use this inner product to define a "dot product" of forms

$$\Omega^p(M; E_{\mathbb{C}}) \otimes \Omega^q(M; E_{\mathbb{C}}) \to \Omega^{p+q}(M; \mathbb{C})$$

which we denote by $\alpha \otimes \beta \mapsto \alpha \cdot \beta$, and which is obtained by wedging the "form part" and applying the inner product $\langle\ ,\ \rangle$ to the coefficients. Analogously, the Lie bracket operation on $u(N) \otimes \mathbb{C}$ (defined by $[a \otimes z, b \otimes w] = [a, b] \otimes zw$) gives rise to a Lie bracket of forms

$$\Omega^p(M; E_{\mathbb{C}}) \otimes \Omega^q(M; E_{\mathbb{C}}) \to \Omega^{p+q}(M; E_{\mathbb{C}}),$$

which we denote by $\alpha \otimes \beta \mapsto [\alpha, \beta]$.

In order to make use of the Riemannian structure (which we haven't used so far), introduce the Hodge star operator

$$* : \Omega^p(M; E_{\mathbb{C}}) \to \Omega^{2\ell-1-p}(M; E_{\mathbb{C}})$$

and define a Hermitian inner product on $\Omega^p(M; E_{\mathbb{C}})$ by

$$\langle \alpha, \beta \rangle = \int_M \alpha \cdot *\beta \in \mathbb{C}.$$

The operator $*$ has the following three properties:

(1) $*$ is an isometry

(2) $*$ is self-adjoint

(3) $(*)^2 = \mathrm{id}_{\Omega^*(M; E_{\mathbb{C}})}$

Properties (2) and (3) above use the fact that M is odd-dimensional. Define the operator $d_A^* : \Omega^p(M; E_{\mathbb{C}}) \to \Omega^{p-1}(M; E_{\mathbb{C}})$ by $d_A^* w = (-1)^p * d_A * w$. It is easy to verify that $\langle d_A \alpha, \beta \rangle = \langle \alpha, d_A^* \beta \rangle$. Denote by T_A the operator $d_A + d_A^*$ on $\Omega^*(M; E_{\mathbb{C}})$. It is well-known that $T_A : \Omega^*(M; E_{\mathbb{C}}) \to \Omega^*(M; E_{\mathbb{C}})$ is a self-adjoint elliptic operator whose domain is the image in $L^2(\Omega^*(M; E_{\mathbb{C}}))$ of the Sobolev space $L^2_1(\Omega^*(M; E_{\mathbb{C}}))$.

To obtain an operator which shares the three properties of $*$ mentioned above, but that also commutes with T_A, we define $\mu : \Omega^p(M; E_{\mathbb{C}}) \to \Omega^{2\ell-1-p}(M; E_{\mathbb{C}})$ by $\mu(w) = i^\ell (-1)^{\frac{p(p+1)}{2}} * w$. Then μ is a self-adjoint isometry, $\mu^2 = \mathrm{id}_{\Omega^*}$ and $T_A \circ \mu = \mu \circ T_A$ on $\Omega^*(M; E_{\mathbb{C}})$. It follows that we have an orthogonal direct sum $\Omega^*(M; E_{\mathbb{C}}) = \Omega^+(M; E_{\mathbb{C}}) \oplus \Omega^-(M; E_{\mathbb{C}})$ where $\Omega^\pm(M; E_{\mathbb{C}})$ is the ± 1- eigenspace of μ. If we take L^2-completions, this decomposition is an orthogonal direct sum of Hilbert spaces. Since T_A commutes with μ, T_A preserves this decomposition. Define functions

$$\Psi_\pm : \Omega^{\mathrm{even}}(M; E_{\mathbb{C}}) \to \Omega^\pm(M; E_{\mathbb{C}})$$

by $\Psi_\pm(w) = \frac{1}{\sqrt{2}}(w \pm \mu(w))$. Clearly $\Psi_\pm$ is an isometry of $\Omega^{\mathrm{even}}(M; E_{\mathbb{C}})$ onto $\Omega^\pm(M; E_{\mathbb{C}})$. The operator $D_A : \Omega^{\mathrm{even}}(M; E_{\mathbb{C}}) \to \Omega^{\mathrm{even}}(M; E_{\mathbb{C}})$ given by $D_A = \Psi_+^{-1} \circ T_A \circ \Psi_+$ is called

the ***signature operator*** for the manifold M. If $w \in \Omega^{2k}(M; E_{\mathbb{C}})$, the explicit formula for D_A is

$$D_A w = i^{\ell}(-1)^{k-1}(*d_A - d_A *)w.$$

We now consider the path of operators D_{A_t}, where A_t is an analytic path of flat connections. To be more precise, we define a path A_t, for $-\varepsilon < t < \varepsilon$, of connections to be analytic if for all $-\varepsilon < t < \varepsilon$, and for all $w \in \Omega^*(M; E_{\mathbb{C}})$,

$$d_{A_t} w = d_{A_0} w + [a(t), w]$$

where $a(t) = a_0 + ta_1 + t^2 a_2 + \ldots$ is a power series in $\Omega^1(M; E)$ which converges in the C^r-norm for all r. Of course the requirement that A_t be flat can be phrased $d_{A_t} \circ d_{A_t} = 0$ for all $t \in (-\varepsilon, \varepsilon)$. To simplify notation, we will write d_t, T_t and D_t instead of d_{A_t}, T_{A_t} and D_{A_t}.

The material of sections 1–4 applies to the deRham complex

$$d_t : \Omega^*(M; E_{\mathbb{C}}) \to \Omega^*(M; E_{\mathbb{C}})$$

and the corresponding operators $T_t = d_t + d_t^*$ to give the subspaces $V_n = \mathcal{H}_n$, the forms B_n, the operators $\widetilde{\delta}_n$ and $\widetilde{\delta}_n^* : \mathcal{H}_n \to \mathcal{H}_n$, etc. We first observe that these spaces and maps respect the grading of $\Omega^*(M; E_{\mathbb{C}})$.

Lemma 5.1. *For each n, the space $\mathcal{H}_n$ breaks up as an orthogonal direct sum*

$$\mathcal{H}_n = \bigoplus_{i=0}^{2\ell-1} \mathcal{H}_n^i$$

where $\mathcal{H}_n^i = \mathcal{H}_n \cap \Omega^i(M; E_{\mathbb{C}})$. Also, $\widetilde{\delta}_n(\mathcal{H}_n^i) \subseteq \mathcal{H}_n^{i+1}$ and $\widetilde{\delta}_n^*(\mathcal{H}_n^i) \subseteq \mathcal{H}_n^{i-1}$.

Proof. We use induction on n. The lemma is clearly true for $\mathcal{H}_0 = \Omega^*(M; E_{\mathbb{C}})$, $\widetilde{\delta}_0 = d_0$ and $\widetilde{\delta}_0^* = d_0^*$.

Now assume the lemma is true for n. Because $\widetilde{\delta}_n$ has degree 1 and $\widetilde{\delta}_n^*$ has degree -1, we know $\ker(\widetilde{\delta}_n) = \bigoplus_{i=1}^{2\ell-1} \ker \widetilde{\delta}_n \cap \Omega^i(M; E_{\mathbb{C}})$ and $\ker(\widetilde{\delta}_n^*) = \bigoplus_{i=1}^{2\ell-1} \ker \widetilde{\delta}_n^* \cap \Omega^i(M; E_{\mathbb{C}})$. Since

$\mathcal{H}_{n+1} = \ker(\widetilde{\delta}_n) \cap \ker(\widetilde{\delta}_n^*)$, it follows that $\mathcal{H}_{n+1} = \overset{2\ell-1}{\underset{i=0}{\oplus}} \mathcal{H}_{n+1}$. To see that $\widetilde{\delta}_{n+1}(\mathcal{H}_{n+1}^i) \subseteq \mathcal{H}_{n+1}^{i+1}$, recall that to define $\widetilde{\delta}_{n+1}(u)$, where $u \in \mathcal{H}_{n+1}^i$, we choose $u_1, \ldots, u_n$ such that $\frac{d^k}{dt^k}|_{t=0} d_t(u + u_1 t + \ldots + u_n t^n) = 0$ for all $k < n$. This equation will still hold if we replace each u_j by its projection in $\Omega^i(M; E_{\mathbb{C}})$ since d_t raises dimension by 1. It follows that $\widetilde{\delta}_{n+1}(u) \in \mathcal{H}_{n+1}^{i+1}$, and similarly that $\widetilde{\delta}_{n+1}^*(u) \in \mathcal{H}_{n+1}^{i-1}$ ∎

In section 4, we showed that $V_n = \mathcal{H}_n = \overset{2\ell-1}{\underset{i=0}{\oplus}} \mathcal{H}_n^i$.

We now turn our attention to the path of operators

$$D_t : \Omega^{\text{even}}(M; E_{\mathbb{C}}) \to \Omega^{\text{even}}(M; E_{\mathbb{C}}).$$

We denote the corresponding spaces and forms constructed in section 1 by $V_n(D)$ and $B_n(D)$ (to distinguish them from the ones just considered which correspond to the path of operators T_t).

Using the fact that $d_t^* = (-1)^p * d_t *$ it is easy to see that $*$ and μ both restrict to isometries between $\mathcal{H}_n^i$ and $\mathcal{H}_n^{2\ell-1-i}$ for all i, n and that for $w \in \mathcal{H}_n^p$, $\widetilde{\delta}_n^*\, w = (-1)^p * \widetilde{\delta}_n * w$. Using the formula $D_t = \Psi^{-1} \circ T_t \circ \Psi$, we deduce that

$$V_n(D) = \Psi^{-1}(V_n) = \bigoplus_{k=0}^{\ell-1} \mathcal{H}_n^{2k}\,.$$

Note. The spaces $\mathcal{G}_n$ and the functions δ_n defined in section 2 also respect the grading; i.e., $\mathcal{G}_n = \overset{2\ell-1}{\underset{i=0}{\oplus}} \mathcal{G}_n^i$ and $\delta_n(\mathcal{G}_n^i) \subseteq \mathcal{G}_n^{i+1}$. Since these spaces and functions were defined without using the metric on M, the spaces $\mathcal{H}_n^i$ (hence V_n and $V_n(D)$) are also independent of the metric (up to isomorphism) since they are canonically isomorphic to $\mathcal{G}_n^i$. Since $V_{n+1}(D)$ is the subspace of $V_n(D)$ on which $B_n(D)$ is degenerate, it follows that the rank of $B_n(D)$ is independent of the metric. In what follows, we will show that the signature of $B_n(D)$ is also independent of the metric.

The following proposition gives a formula for the form $B_n(D)$ on $V_n(D)$ in terms of $\widetilde{\delta}_n$ and the Hodge star.

Proposition 5.2. *If $v \in \mathcal{H}_n^{2k}$ and $w \in \overset{\ell-1}{\underset{i=0}{\oplus}} \mathcal{H}_n^{2i}$ then*

$$B_n(D)(v,w) = i^\ell(-1)^{k-1}\langle(*\,\widetilde{\delta}_n - \widetilde{\delta}_n\,*)v, w\rangle.$$

Proof. In the notation of section 1, write

$$v = \sum_{\alpha \in A_n} c_\alpha\, \varphi_\alpha(0)$$

and let $v(t) = \sum c_\alpha\, \varphi_\alpha(t)$. (Recall that $V_n(D) = \underset{\alpha\in A_n}{\operatorname{span}}\{\varphi_\alpha(0)\}$.) From section 1, $B_n(D)(v,w) = \frac{d^n}{dt^n}\big|_{t=0}\langle D_t v(t), w\rangle$. By definition of A_n, $\frac{d^k}{dt^k}\big|_{t=0} D_t v(t) = 0$ for all $k < n$. Since $\operatorname{im}(*d_t) \perp \operatorname{im}(d_t)$ for all t, it follows by Lemma 4.2 that for all $k < n$, $\frac{d^k}{dt^k}\big|_{t=0} d_t v(t) = 0 = \frac{d^k}{dt^k}\big|_{t=0} d_t * v(t)$. Therefore $\widetilde{\delta}_n(v) = pr_{\mathcal{H}_n}\frac{d^n}{dt^n}\big|_{t=0} d_t v(t)$ and $\widetilde{\delta}_n(*v) = pr_{\mathcal{H}_n}\frac{d^n}{dt^n}\big|_{t=0} d_t * v(t)$. These equations and the definition of $B_n(D)$ imply that $B_n(D)(v,w) = i^\ell(-1)^{k-1}\langle(*\,\widetilde{\delta}_n - \widetilde{\delta}_n\,*)v, w\rangle$. ∎

The essence of this proposition is: To calculate the n^{th} derivatives of those eigenvalues of D_t which vanish (at $t = 0$) to order $n-1$, we can simply diagonalize the signature operator on the even part of the chain complex $(\mathcal{H}_n, \widetilde{\delta}_n)$. The kernel of this new signature operator is isomorphic to the even part of the cohomology of $(\mathcal{H}_n, \widetilde{\delta}_n)$, which is just $\overset{\ell-1}{\underset{i=0}{\oplus}} \mathcal{H}_{n+1}^{2i}$.

Before showing that the signature of the forms $B_n(D)$ are invariant of the metric on M, we make some further observations on the path of operators $d_t : \Omega^*(M; E_{\mathbb{C}}) \to \Omega^*(M; E_{\mathbb{C}})$. In section 2, we defined the subspaces $\mathcal{Z}_n$ and $\mathcal{B}_n$ arising from this type of setup. It is easy to see that these subspaces respect the grading; hence we write

$$\mathcal{Z}_n = \overset{2\ell-1}{\underset{i=1}{\oplus}} \mathcal{Z}_n^i \quad \text{and} \quad \mathcal{B}_n = \overset{2\ell-1}{\underset{i=0}{\oplus}} \mathcal{B}_n^i$$

where $\mathcal{Z}_n^i = \mathcal{Z}_n \cap \Omega^i(M; E_{\mathbb{C}})$ and $\mathcal{B}_n^i = (\mathcal{B}_n \cap \Omega^i(M; E_{\mathbb{C}})$. Also $\mathcal{G}_n = \overset{2\ell-1}{\underset{i=0}{\oplus}} \mathcal{G}_n^i$ where $\mathcal{G}_n^i = \mathcal{Z}_n^i \,/\, \mathcal{B}_n^i$.

Proposition 5.3. *Consider the pairing $\Omega^k(M; E_{\mathbb{C}}) \times \Omega^{2\ell-1-k}(M; E_{\mathbb{C}}) \to \mathbb{C}$ defined by $(\alpha, \beta) \mapsto \int_M \alpha \cdot \beta$, which is $\mathbb{C}$-linear in the first factor and conjugate-linear in the second.*

This pairing induces a well-defined pairing

$$\mathcal{G}_n^k \times \mathcal{G}_n^{2\ell-1-k} \to \mathbb{C} \quad \text{for all } n, k.$$

Proof. Since $\int\limits_M \alpha \cdot \beta = \overline{\int\limits_M \beta \cdot \alpha}$, it will suffice to show that if $\alpha \in \mathcal{B}_n^k$ and $\beta \in \mathcal{Z}^{2\ell-1-k}$, then $\int\limits_M \alpha \cdot \beta = 0$. This reduces to showing that if $\alpha \in \text{Range}(d_0)$ or $\alpha \in \text{Range}(T_{n-1})$ then $\int \alpha \cdot \beta = 0$. If $\alpha \in \text{Range}(d_0)$, i.e., if $\alpha = d_0\gamma$, then $\int_M \alpha \cdot \beta = \int d_0\gamma \cdot \beta = \pm \int \gamma \cdot d_0\beta = 0$ since $\mathcal{Z}_n \subseteq \ker(d_0)$. If $\alpha \in \text{Range}(T_{n-1})$ then we may find a series $\gamma(t)$ in $\Omega^{k-1}(M; E_{\mathbb{C}})$ such that $\frac{d^i}{dt^i}\big|_{t=0} d_t\gamma(t) = 0$ for all $i < n-1$, and $\frac{d^{n-1}}{dt^{n-1}}\big|_{t=0} d_t\gamma(t) = \alpha$. Since $\beta \in \mathcal{Z}_n^{2\ell-1-k}$, we may find a series $\beta(t)$ in $\Omega^{2\ell-1-k}(M; E_{\mathbb{C}})$ such that $\beta(0) = \beta$ and $\frac{d^i}{dt^i}\big|_{t=0} d_t\beta(t) = 0$ for all $i < n$. We then compute

$$\begin{aligned}
\int\limits_M \alpha \cdot \beta &= \frac{d^{n-1}}{dt^{n-1}}\Big|_{t=0} \int\limits_M d_t\gamma(t) \cdot \beta(t) \\
&= \pm \frac{d^{n-1}}{dt^{n-1}}\Big|_{t=0} \int\limits_M \gamma(t) \cdot d_t\beta(t) \\
&= 0,
\end{aligned}$$

completing the proof. ∎

We now define a Hermitian form $Q_n : \mathcal{G}_n^{\ell-1} \times \mathcal{G}_n^{\ell-1} \to \mathbb{C}$ by

$$Q_n(v, w) = \begin{cases} i(-1)^{\frac{\ell+1}{2}} \int\limits_M \delta_n v \cdot w & \text{if } \ell \text{ is odd} \\ (-1)^{\frac{\ell}{2}} \int\limits_M \delta_n v \cdot w & \text{if } \ell \text{ is even} \end{cases}$$

(Recall $\delta_n : \mathcal{G}_n^p \to \mathcal{G}_n^{p+1}$ was defined in section 2.) Note that Q_n is defined without reference to the metric on M.

Theorem 5.4. *The forms $B_n(D)$ and Q_n have the same signatures.*

Proof. First, consider the case in which ℓ is odd. Recall $V_n(D) = \overset{\ell-1}{\underset{k=0}{\oplus}} \mathcal{H}_n^{2k}$, and that $\{\mathcal{H}_n^i\}_{0\le i\le 2\ell-1}$ is a cochain complex with boundary $\widetilde{\delta}_n$. Define the operator $D_{(n)} : V_n(D)\to V_n(D)$ by $D_{(n)}v = i^\ell(-1)^{k-1}(*\,\widetilde{\delta}_n - \widetilde{\delta}_n\,*)v$, so $B_n(D)(v,w) = \langle D_{(n)}v, w\rangle$. Write $V_n(D) = X\oplus Y\oplus W$ where

$$X = \left(\underset{0\le k\le\frac{\ell-3}{2}}{\oplus} \mathcal{H}_n^{2k}\right) \oplus \mathrm{im}(\widetilde{\delta}_n : \mathcal{H}_n^{\ell-2}\to\mathcal{H}_n^{\ell-1})$$

$$Y = \underset{0\le k\le\frac{\ell-3}{2}}{\oplus} \mathcal{H}^{\ell+1+2k}$$

$$W = \ker(\widetilde{\delta}_n\,* : \mathcal{H}_n^{\ell-1}\to\mathcal{H}_n^{\ell+1})$$

Conceptually, X = all even forms in $\mathcal{H}_n$ of dimension less than $\ell-1$ plus the "exact" forms of dimension $\ell-1$, Y = all even forms of dimension greater than $\ell-1$ and W = the "coclosed" $\ell-1$ forms. (The words "exact" and "coclosed" refer to the operator $\widetilde{\delta}_n$.)

We omit the verifications of the following facts, which are routine: $D_{(n)}(X)\subseteq Y$, $D_{(n)}(Y)\subseteq X$, $(D_{(n)}|X)^* = (D_{(n)}|Y)$, and $D_{(n)}(W)\subseteq W$. It follows immediately that $B_n(D)$ has the same signature as $B_n(D)|W$. For $v,w\in W$, note that

$$B_n(D)(v,w) = \langle D_{(n)}v,w\rangle = i^\ell(-1)^\ell\langle *\,\widetilde{\delta}_n\,v,w\rangle$$

$$-i^\ell\langle\widetilde{\delta}_n\,v,*w\rangle = i(-1)^{\frac{\ell+1}{2}}\int\widetilde{\delta}_n\,v\cdot w = Q_n(v,w),$$

where in the last step we are thinking of v and w as representing classes in $\mathcal{G}_n$.

Since $\mathrm{im}(\widetilde{\delta}_n : \mathcal{H}_n^{\ell-2}\to\mathcal{H}_n^{\ell-1})$ is contained in the degeneracy of Q_n, it follows that Q_n, $B_n(D)|W$, and $B_n(D)$ all have the same signature, completing the proof for ℓ odd.

If ℓ is even the method of proof is the same, only the space W is replaced by the space

$$U = \ker(\widetilde{\delta}_n : \mathcal{H}_n^\ell\to\mathcal{H}_n^{\ell+1}).$$

It follows that $B_n(D)$ has the same signature as the form $(v,w)\mapsto(-1)^{\frac{\ell}{2}}\int\widetilde{\delta}_n *v\cdot *w$ or U which, since $*$ is an isometry, has the same signature as Q_n on $\mathcal{G}_u^{\ell-1}$. This completes the proof of Theorem 5.4. ∎

6. Massey products

Thus far we have demonstrated the existence of a sequence of chain complexes $\{(\mathcal{G}_n^i, \delta_n)\}$ such that $(\mathcal{G}_0^i, \delta_0)$ is just the deRham complex $(\Omega^i(M; E_{\mathbb{C}}), d_0)$ and such that the chain groups of the complex $(\mathcal{G}_n^i, \delta_n)$ are the homology groups of the previous complex $(\mathcal{G}_{n-1}^i, \delta_{n-1})$. Furthermore, the rank of $\mathcal{G}_n^{\text{even}} = \bigoplus_{k=0}^{\ell-1} \mathcal{G}_n^{2k}$ is the number of eigenvalues of the signature operator D_t which vanish to order $n-1$ at $t=0$. Finally, of those eigenvalues which vanish to order $n-1$, the difference between the number of eigenvalues whose n^{th} derivative is positive and the number whose n^{th} derivative is negative is given by the signature of the form $Q_n : \mathcal{G}_n^{\ell-1} \times \mathcal{G}_n^{\ell-1} \to \mathbb{C}$ which is independent of the metric on M.

In this section we will show that for a path $d_t : \Omega^*(M; E_{\mathbb{C}}) \to \Omega^*(M; E_{\mathbb{C}})$ of flat connections, the functions δ_n defined in section 2 can be identified with certain higher Massey products which are well-known to be invariants of the homotopy type of M.

The deRham complex $\{\Omega^*(M; E_{\mathbb{C}}), d_0\}$ is a differential graded Lie algebra and it is in this context that we will now work.

Definition. A ***graded Lie algebra*** (GLA) is a sequence $\{L^n\}_{n=0}^{\infty}$ of complex vector spaces together with a bilinear operation $[\ ,\ ] : L^m \times L^n \to L^{m+n}$ satisfying:

(1) $[x, y] + (-1)^{mn}[y, x] = 0$

(2) $[x, [y, z]] = [[x, y], z] + (-1)^{mn}[y, [x, z]]$

where we are assuming $x \in L^m$ and $y \in L^n$. (2) is just the graded version of the Jacobi identity. We write $L = \bigoplus_{n=0}^{\infty} L^n$, and extend $[\ ,\ ]$ linearly over L.

A ***differential graded Lie algebra*** (DGLA) is a GLA equipped with a differential

$$d : L^m \to L^{m+1}$$

for all m which satisfies $d[x, y] = [dx, y] + (-1)^{|x|}[x, dy]$. ($|x|$ is defined by $x \in L^{|x|}$.)

Assume (L, d) is a DGLA. For each $a \in L^1$, we define $d_a : L^m \to L^{m+1}$ by $d_a x =$

$dx + [a,x]$. The Jacobi identity implies that (L, d_a) is also a DGLA. We call the DGLA (L,d) ***flat*** if $d \circ d = 0$. If (L,d) is flat then (L, d_a) is flat if and only if $da + \frac{1}{2}[a,a] = 0$.

Note that if A is a flat connection, then the deRham complex $(\Omega^*(M; E_{\mathbb{C}}), d_A)$ introduced in section 5 is a flat DGLA. It is a well-known fact that the exterior derivatives $d_{A'}$, coming from other connections A' on P, are precisely those differentials of the form $d_{A'}x = d_A x + [a,x]$ where $a \in \Omega^1(M; E_{\mathbb{C}})$. We say the form a is ***flat*** with respect to a fixed flat connection A if $d_A a + \frac{1}{2}[a,a] = 0$.

Let (L,d) be a flat DGLA throughout the rest of the section. Let $L[[t]] \equiv \bigoplus_{n=0}^{\infty} L^n[[t]]$ denote the space of formal power series with coefficients in L. By defining $[xt^n, yt^m] = [x,y]t^{n+m}$ and $d(xt^n) = d(x)t^n$ we give $(L[[t]], d)$ the structure of a flat DGLA. If $a(t) = \sum_{i=1}^{\infty} a_i t^i \in L^1[[t]]$ then we may define $d_{a(t)} : L[[t]] \to L[[t]]$ as before by $d_{a(t)} x(t) = dx(t) + [a(t), x(t)]$. Note that $a(t)$ is flat if and only if the following sequence of equations holds:

$$
\begin{aligned}
da_1 &= 0 \\
da_2 &= -\frac{1}{2}[a_1, a_1] \\
da_3 &= -\frac{1}{2}([a_1, a_2] + [a_2, a_1]) \\
&\vdots \\
da_n &= -\frac{1}{2}\sum_{i=1}^{n-1}[a_i, a_{n-i}]
\end{aligned}
$$

Given a flat $a(t) \in L^1[[t]]$, with vanishing constant term we now define a sequence of relations T_n on L. These relations will coincide, in the case where $a(t)$ is convergent for $t \in (-\varepsilon, \varepsilon)$, with the relations defined in section 2 for the operator $d_t x = dx + [a(t), x]$.

Define $uT_n v$ if and only if there exist $u_1, \ldots, u_{n-1} \in L$ such that

$$d_{a(t)}(u + u_1 t + \ldots + u_{n-1}t^{n-1}) = vt^n + w(t)t^{n+1}$$

for some $w(t) \in L[[t]]$.

If we define $\mathcal{Z}_n$, $\mathcal{B}_n$, $\mathcal{G}_n = \mathcal{Z}_n / \mathcal{B}_n$, and $\delta_n : \mathcal{G}_n \to \mathcal{G}_n$ just as in section 2, (with δ_n induced by $n!T_n$) then the proof of Theorem 2.1 implies that $(\mathcal{G}_n, \delta_n)$ is a sequence of well-defined chain complexes with the chain groups of each complex equal to the homology of the one before, i.e., $\mathcal{G}_{n+1} = \ker \delta_n / \operatorname{im} \delta_n$. Next, we will show that in the context of a flat DGLA the operator δ_n can be expressed in terms of Massey products as defined by Retakh in [Re].

We begin by reviewing Retakh's definition of higher Massey products in a flat DGLA. We define a ***multi-index I*** with $|I| = n \geq 1$ to be a set of integers $I = \{i_1, \ldots, i_n\}$ such that $1 \leq i_1 < i_2 < \ldots < i_n$. A ***submulti-index of I*** is a non-empty subset of I. A ***proper pair*** of submulti-indices of I is an ordered pair (J, K) of submulti-indices of I, where we write

$$J = \{j_1 < \ldots < j_r\} \qquad \text{and} \qquad K = \{k_1 < \ldots < k_s\}$$

which satisfy

(1) $J \cap K = \emptyset$

(2) $J \cup K = I$

(3) $J \neq \emptyset \neq K$

(4) $j_1 < k_1$ (i.e., $j_1 = i_1$)

The set of all proper pairs of submulti-indices of I is denoted $PP(I)$.

For each homogeneous element $x \in L$, let $\overline{x} = (-1)^{|x|+1} x$. Let $x_1, \ldots, x_n$ be an ordered n- tuple of homogeneous cocycles in L. A ***Massey system*** $\mathcal{M}$ for $(x_1, \ldots, x_n)$ consists of a homogeneous element $u_I \in L$ for each multi-index $I \subset \{1, 2, \ldots, n\}$ satisfying the following two properties:

(1) $u_{\{i\}} = x_i$ for each $i \in \{1, \ldots, n\}$

(2) For $|I| > 1$,

$$du_I = \sum_{(J,K) \in PP(I)} (-1)^{\varepsilon(J,K)} [\overline{u}_J, u_K]$$

where $\varepsilon(J, K) = \sum_{(j,k) \in J \times K : k < j} (|x_j| + 1)(|x_k| + 1)$.

Given a Massey system $\mathcal{M} = \{u_I\}$ for $\{x_1, \ldots, x_n\}$, the corresponding Massey product is defined by

$$\widetilde{Q}_{\mathcal{M}}(x_1, \ldots, x_n) = \sum_{(J,K) \in PP\{1,\ldots,n\}} (-1)^{\varepsilon(J,K)} [\overline{u}_J, u_K]$$

In practice, to construct a Massey system one constructs the elements u_I inductively according to $|I|$. To begin, if $1 \leq i_1 < i_2 \leq n$, one selects a homogeneous element u_{i_1,i_2} such that

$$du_{i_1,i_2} = [\overline{x}_{i_1}, x_{i_2}].$$

To accomplish this step, it must be true that $[x_{i_1}, x_{i_2}] = 0$ in $H^*(L)$. Of course, this condition only determines u_{i_1,i_2} up to addition of a 1-cocycle. Having defined all the u_{i_1,i_2}s, one next must choose u_{i_1,i_2,i_3} (for $1 \leq i_1 < i_2 < i_3 \leq n$) satisfying

$$du_{i_1,i_2,i_3} = [\overline{u}_1, u_{23}] + [\overline{u}_{12}, u_3] + (-1)^{(|x_2|+1)(|x_3|+1)}[\overline{u}_{13}, u_2],$$

et cetera.

It is a routine exercise (using the Jacobi identity) to show that once one has defined the elements u_J for all $|J| < m$ then, if $|I| = m$, $\sum\limits_{(J,K) \in PP(I)} (-1)^{\varepsilon(J,K)} [\overline{u}_J, u_K]$ is a cocycle of degree $(\sum\limits_{i \in I} |x_i|) - |I| + 1$. Hence in order to define u_I, this cocycle must be a coboundary.

As a special case of the above "routine exercise" we have the following proposition.

Proposition 6.1. *If $\mathcal{M}$ is a Massey system for $\{x_1, \ldots, x_n\}$, then $\widetilde{Q}_{\mathcal{M}}(x_1, \ldots, x_n)$ is a cocycle of degree $(\sum\limits_{1 \leq i \leq n} |x_i|) - n + 1$.*

Definition. Let $Q_{\mathcal{M}}(x_1, \ldots, x_n)$ denote the cohomology class in $H^*(L)$ represented by $\widetilde{Q}_{\mathcal{M}}(x_1, \ldots, x_n)$.

In general $Q_{\mathcal{M}}(x_1, \ldots, x_n)$ depends on which Massey system $\mathcal{M}$ was chosen. Note that a Massey system for $(x_1, \ldots, x_n)$ contains Massey systems for all proper subsets of $(x_1, \ldots, x_n)$ for which the corresponding Massey products vanish. Thus the Massey

product cannot be defined for an arbitrary n-tuple of cocycles. The second order Massey product is, however, well-defined for every pair (x_1, x_2) of cocycles; it is simply the "cup product" $[\overline{x}_1, x_2]$.

Proposition 6.2. *Let the power series $a(t) = \sum_{i=1}^{\infty} a_i t^i$ in $L^1[[t]]$ be flat in the sense that $d_{a(t)} \circ d_{a(t)} = 0$, as discussed earlier. For all multi-indices I, define $u_I = (-1)^{n+1} n! a_n$, where $n = |I|$. Then*

(1) $u_{\{i\}} = a_1$ for all $i \in \mathbb{Z}_+$

and

(2) $du_I = \sum_{(J,K) \in PP(I)} (-1)^{\varepsilon(J,K)} [\overline{u}_J, u_K]$ for all $i \subseteq \mathbb{Z}_+$.

In words, this proposition states that the elements u_I, as defined in the proposition, contain a Massey system for the ordered n-tuple $x_1 = x_2 = \ldots = x_n = a_1$ for every n, and all of the corresponding Massey products vanish.

Proof. Set $c_n = (-1)^{n+1} n! a_n$ for each $n \geq 1$. Assume $|I| = n$. Compute $du_I = dc_n = (-1)^{n+1} n! da_n = (-1)^{n+1} n! (-\frac{1}{2}) \sum_{i=1}^{n-1} [a_i, a_{n-i}] = \frac{1}{2} \sum_{i=1}^{n-1} \binom{n}{i} [c_i, c_{n-i}] = \sum_{(J,K) \in PP(I)} [u_J, u_K]$, where the last step follows by an easy combinatorial argument. Because $u_I \in L^1$ for all I, it follows that $\overline{u}_I = u_I$ and $(-1)^{\varepsilon(J,K)} = 1$, so the proposition follows. ∎

Continue to assume that $a(t) = \sum_{i=1}^{\infty} a_i t^i$ is a flat series in $L^1[[t]]$. Let u be a homogeneous cocycle (of any degree) and note that a_1 is also a cocycle since $a(t)$ is flat. We will now show that $u T_n v$ if and only if $\widetilde{Q}_{\mathcal{M}}(u, \underbrace{a_1, \ldots, a_1}_{n}) = (-1)^{n+1} n! v$ for some Massey system $\mathcal{M}$ which is $a(t)$-compatible in the following sense:

Definition. Suppose $n \geq 1$, $a(t)$ is a flat series in $L^1[[t]]$, u is a homogeneous cocycle,

$\mathcal{M} = \{u_I\}$ is a Massey system for $(u, \underbrace{a_1, \ldots, a_1}_{n})$. Then we say $\mathcal{M}$ is $a(t)$-compatible if and only if

(1) if $1 \notin I$, then $u_I = (-1)^{n+1} n! a_n$ where $n = |I|$

and

(2) if $1 \in I_1 \cap I_2$ and $|I_1| = |I_2|$, then $u_{I_1} = u_{I_2}$.

More informally, this definition says that for $\mathcal{M}$ to be $a(t)$-compatible: (1) $\mathcal{M}$ must make use of the Massey systems contained in $a(t)$ to the largest extent possible; and (2) $\mathcal{M}$ must have as much symmetry as possible.

Proposition 6.3. *Let $d_t = d_{a(t)} : L \to L$, where $a(t) \in L^1[[t]]$ is flat. Then $uT_n v$ if and only if $\widetilde{Q}_{\mathcal{M}}(u, \underbrace{a_1, \ldots, a_1}_{n}) = (-1)^{n+1} n! v$ for some $a(t)$-compatible Massey system $\mathcal{M}$.*

Proof. We will prove the ($\Rightarrow$) direction. The proof of ($\Leftarrow$) is just a reversal of the argument and will be omitted. Recall that $uT_n v$ means there exists $u_1, \ldots, u_{n-1} \in L$ such that

$$d_t(u + u_1 t + \ldots + u_{n-1} t^{n-1}) = v t^n + w(t) t^{n+1}.$$

This implies that

$$\begin{aligned} &du = 0 \\ &du_1 + [a_1, u] = 0 \\ &\vdots \\ &du_{n-1} + [a_1, u_{n-2}] + \ldots + [a_{n-1}, u] = 0. \end{aligned}$$

It follows that if we set $u_I = (-1)^{n+1} n! a_n$ (where $n = |I|$) if $1 \notin I$ and $u_I = (-1)^n n! u_n$ (where $n = |I| - 1$) if $1 \in I$ then $\mathcal{M} = \{u_I\}_I$ s an $a(t)$-compatible Massey system and

$\widetilde{Q}(u, \underbrace{a_1, \ldots, a_1}_{n}) = (-1)^{n+1} n! v$. To verify this last fact, note that

$$\widetilde{Q}_{\mathcal{M}}(u, a_1, \ldots, a_1) = \sum_{(J,K) \in PP\{1,\ldots,n+1\}} (-1)^{\varepsilon(J,K)} [\overline{u}_J, u_K].$$

$\varepsilon(J, K) = 1$ because $|a_1| = 1$, and

$$\overline{u}_J = (-1)^{|u_J|+1} u_J = (-1)^{|u_J|+1} u_J = (-1)^{|u|+1} u_J,$$

so

$$\begin{aligned}
\widetilde{Q}_{\mathcal{M}}(u, a_1, \ldots, a_1) &= (-1)^{|u|+1} \sum_{(J,K) \in PP\{1,\ldots,n+1\}} [u_J, u_K] \\
&= (-1)^{|u|+1} \sum_{i=0}^{n-1} \binom{n}{i} [(-1)^i (i!) u_i, (-1)^{n-i+1} (n-i)! a_{n-1} \\
&= n! (-1)^{n+1} \sum_{i=0}^{n-1} [a_{n-i}, u_i] \\
&= (-1)^{n+1} n! v.
\end{aligned}$$

We omit the routine verification that $\mathcal{M}$ is an $a(t)$-compatible Massey system. This completes the proof of the $(\Rightarrow)$ direction. ∎

Recall from section 2 that $\tau_n : \mathcal{G}_n \to \mathcal{G}_n$ is just the well-defined map induced by the relation T_n on V, and $\delta_n = n! \tau_n$. Hence for $[u] \in \mathcal{G}_n^i$, we may write

$$\delta_n[u] = (-1)^{n+1} [\widetilde{Q}_{\mathcal{M}}(u, \underbrace{a_1, \ldots, a_1}_{n})] \in \mathcal{G}_n^{i+1}$$

Note the following consequence of what we have proved: the normally ill-defined Massey product

$$u \mapsto Q_{\mathcal{M}}(u, \underbrace{a_1, \ldots, a_1}_{n})$$

becomes well-defined when we insist that $\mathcal{M}$ be $a(t)$-compatible, and when we think of the domain and range of the function as being the subquotients $\mathcal{G}_n^i$ of $H^i(M; E_{\mathbb{C}})$ and $\mathcal{G}_n^{i+1}$ of $H^{i+1}(M; E_{\mathbb{C}})$.

7. General bundles

In the previous two sections we chose the bundle E to be the complexification of the adjoint bundle associated to some principal bundle.

This was done for simplicity and we indicate now what modifications need to be made to handle the case when E is a general vector bundle with a path of flat connections.

Let $P \to M$ be a principal G bundle for some compact lie group G with Lie algebra $\mathcal{G}$, and let $\rho : G \to U(W)$ be a unitary representation of G onto a hermitian vector space W, defining a vector bundle $E = P \times_\rho W$. Let adP denote the adjoint bundle of lie algebras, $adP = P \times_{ad} \mathcal{G}$. An analytic path of flat connections on P defines an analytic path of operators $d_t^E : \Omega^*(M;E) \to \Omega^*(M;E)$, as well as an analytic path of operators $d_t^{ad} : \Omega^*(M;adP) \to \Omega^*(M;adP)$. Then $a(t) = d_t^{ad} - d_0^{ad}$ is an analytic path in $\Omega^1(M;adP)$ and $d_t^E = d_0^E + \rho_*(a(t))$ where $\rho_* : \mathcal{G} \to gl(W)$ is the differential of ρ. For simplicity of notation we assume that $\mathcal{G}$ acts on W on the *right*.

The differential of ρ endows $\Omega^*(M;E)$ with the structure of a module over the DGLA $\Omega^*(M;adP)$. Thus to extend the definitions of Massey products we need a "hybrid" generalization which takes this action into account. This is done as follows.

If $x_1 \in \Omega^*(M;E)$ is a d_0^E-cocycle and $x_i \in \Omega^*(M;adP),\ i = 2, \cdots n$ are d_0^{ad}-cocycles, we define Massey systems as before, with the caveats:

1. If $1 \in I$, then $u_I \in \Omega^*(M;E)$, if $1 \notin I$, then $u_I \in \Omega^*(M;adP)$.
2. If $1 \notin J \cup K$, then $[\bar{u}_J, u_K]$ has the same meaning as before. If $1 \in J$, then $[\bar{u}_J, u_K]$ means $\bar{u}_J \rho_*(u_K)$, i.e. the action of u_K on $\bar{u}_J$.
3. In the equation
$$du_I = \sum_{(J,K) \in PP(I)} (-1)^{\varepsilon(J,K)} [\overline{u}_J, u_K],$$
one should interpret the operator d as d_0^E or d_0^{ad} according to whether or not $1 \in I$.

With this definition, one can define Massey products

$$\mathcal{Q}_{\mathcal{M}}(e, y_1, \cdots, y_n) \in H^*(M;E)_{A(0)}$$

of cocycles $e \in \Omega^*(M;E)$ and $y_i \in \Omega^*(M; adP)$.

The definition of $a(t)$-compatible Massey systems for $\mathcal{Q}(u, a_1, \cdots, a_1)$ is similar: one assumes that $u_I = (-1)^{n+1} n! a_n$ if $1 \notin I$, and if $1 \in I_1 \cap I_2$ and $|I_1| = |I_2|$, then $u_{I_1} = u_{I_2}$. Then the theorem of the previous section continues to hold with these definitions.

Finally, to define the bilinear forms Q_n, one need only use the "dot" product coming from a G-invariant fiber metric on the bundle E. With these definitions, all the results of Section 5 and 6 continue to hold for E a general vector bundle.

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